The Active Bijection in Graphs, Hyperplane Arrangements, and Oriented Matroids 1. The Fully Optimal Basis of a Bounded Region. Erratum

Emeric Gioan* Michel Las Vergnas†

December 5, 2010

In [1], Proposition 5.1 and Theorem 5.3 do not hold as stated. In both, the hypothesis that $p < f$ are the first two elements of $B_{\min}$ should be replaced by the hypothesis that $p < f$ are the first two elements of E. Under this new hypothesis, proofs are valid without change[1]. However, to avoid a possible confusion with the notation $B_{\min} = \{p, f, \ldots\}_<$ used throughout the paper, we should rather write $E = \{e_1, e_2, \ldots\}_<$.

Proposition 5.1 *Let M be an ordered matroid on a set $E = \{e_1, e_2, \ldots\}_<$. A basis B of M is internal and uniactive if and only if $(E \setminus B) \cup \{e_1\} \setminus \{e_2\}$ is internal and uniactive in M^*.*

Theorem 5.3 *Let M be a bounded acyclic ordered oriented matroid on a set $E = \{e_1, e_2, \ldots\}_<$. We have*

$$\alpha(-_{e_1}M^*) = \big(E \setminus \alpha(M)\big) \cup \{e_1\} \setminus \{e_2\}.$$

The duality property in Theorem 5.3 is called the *active duality*. In the last part of Section 5, when comparing active duality to linear programming duality, it is implicitly assumed that $p = e_1$ and $f = e_2$, implying that $\{e_1, e_2\}$ is independent.

[1] E. Gioan, M. Las Vergnas, *The active bijection in graphs, hyperplane arrangements, and oriented matroids 1. The fully optimal basis of a bounded region*, European Journal of Combinatorics **30 (8)** (2009), 1868–1886.

*C.N.R.S., Montpellier
†C.N.R.S., Paris

[1] Independently, in line 10 of the proof of Proposition 5.1, instead of $B' - f$ read $(E \setminus B') \setminus \{f\}$.

European Journal of Combinatorics 30 (2009) 1868–1886

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The active bijection in graphs, hyperplane arrangements, and oriented matroids, 1: The fully optimal basis of a bounded region

Emeric Gioan [a], Michel Las Vergnas [b]

[a] CNRS - LIRMM, 161 rue Ada 34392 Montpellier Cedex 5, France

[b] CNRS - Université Pierre et Marie Curie (Paris 6), case 189 - Combinatoire et Optimisation, 4 place Jussieu, 75252 Paris Cedex 05, France

ARTICLE INFO

Article history:
Available online 15 February 2009

ABSTRACT

The present paper is the first in a series of four dealing with a mapping, introduced by the present authors, from orientations to spanning trees in graphs, from regions to simplices in real hyperplane arrangements, from reorientations to bases in oriented matroids (in order of increasing generality). This mapping is actually defined for ordered oriented matroids. We call it the active orientation-to-basis mapping, in reference to an extensive use of activities, a notion depending on a linear ordering, first introduced by W.T. Tutte for spanning trees in graphs. The active mapping, which preserves activities, can be considered as a bijective generalization of a polynomial identity relating two expressions – one in terms of activities of reorientations, and the other in terms of activities of bases – of the Tutte polynomial of a graph, a hyperplane arrangement or an oriented matroid. Specializations include bijective versions of well-known enumerative results related to the counting of acyclic orientations in graphs or of regions in hyperplane arrangements. Other interesting features of the active mapping are links established between linear programming and the Tutte polynomial.

We consider here the bounded case of the active mapping, where bounded corresponds to bipolar orientations in the case of graphs, and refers to bounded regions in the case of real hyperplane arrangements, or of oriented matroids. In terms of activities, this is the uniactive internal case. We introduce fully optimal bases, simply defined in terms of signs, strengthening optimal bases of linear programming. An optimal basis is associated with one

E-mail addresses: Emeric.Gioan@lirmm.fr (E. Gioan), mlv@math.jussieu.fr (M. Las Vergnas).

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2008.12.013

flat with a maximality property, whereas a fully optimal basis is equivalent to a complete flag of flats, each with a maximality property. The main results of the paper are that a bounded region has a unique fully optimal basis, and that, up to negating all signs, fully optimal bases provide a bijection between bounded regions and uniactive internal bases. In the bounded case, up to negating all signs, the active mapping is a bijection.

1. Introduction

The present paper is the first in a series of four dealing with a mapping from orientations to spanning trees in graphs, from regions to simplices[1] in real hyperplane arrangements, from reorientations to bases in oriented matroids (in order of increasing generality[2]). This mapping is actually defined for ordered oriented matroids. It has been introduced by the present authors for graphs and different other particular cases in previous papers [7–9]. We present it here for the first time in full extent and generality. We call it the *active orientation-to-basis mapping*, in reference to an extensive use of activities, a notion depending on a linear ordering of the ground set, first introduced by W.T. Tutte for spanning trees in graphs [22]. We will show later that the active mapping has bijective formulations.

With the exception of [10], partly written in terms of real hyperplane arrangements, the papers in the series are written in the language of oriented matroids, which is both general and convenient for our purpose. We have tried to make them reasonably self-contained. Readers unacquainted with oriented matroid theory can get an understanding of the active mapping in the context of graphs [8] or of real hyperplane arrangements [9]. A general reference for oriented matroids is [1] (see also [2]). For the convenience of the reader, Section 2 recalls the needed background.

The active mapping is a bijective generalization of a polynomial identity relating two expressions in terms of activities of the Tutte polynomial of an oriented matroid. Specializations include well-known results such as the number of acyclic orientations of a graph as the evaluation $\chi(-1)$ of its chromatic polynomial, or the number of regions of a real hyperplane arrangement as the evaluation $p(-1)$ of the Poincaré polynomial of its lattice [15,20,23,24]. Other interesting features of the active mapping are links established between linear programming and the Tutte polynomial.

The *Tutte polynomial* of a matroid is a 2-variable polynomial with non-negative integer coefficients, first considered by W.T. Tutte for graphs [22], then also for matroids by H.H. Crapo [5]. Up to simple algebraic transformations, the Tutte polynomial of a matroid is equivalent to its *rank-generating function*, i.e. to the 2-variable generating function of cardinality and rank of subsets of elements. The Tutte polynomial is a fundamental tool for many numerical invariants in graphs and matroids, and has numerous applications (see for instance [4]).

Let M be a matroid on a linearly ordered set of elements E. By a classical theorem [22,5], we have

$$t(M; x, y) = \sum_{i,j=0,1,\dots} b_{i,j} x^i y^j$$

where $b_{i,j}$ is the number of bases of M such that i basis elements are the smallest in their fundamental cocircuit and j non-basis elements are the smallest in their fundamental circuit. The parameters i and j for a given basis are classically called its *internal* and *external activities*.

[1] In the present paper, a 'simplex' is not a simplicial region, but a basis of hyperplanes, that is an inclusion maximal set of independent hyperplanes.

[2] Regions in real hyperplane arrangements generalize acyclic directed graphs. To generalize all directed graphs, arrangements of signed real hyperplanes will be considered.

Let M be an oriented matroid. M. Las Vergnas has shown in [18] that

$$t(M; x, y) = \sum_{i,j=0,1,\dots} o_{i,j} 2^{-i-j} x^i y^j$$

where $o_{i,j}$ is the number of subsets A of E such that i elements of E are the smallest in some positive cocircuit of the reorientation $-_A M$ of M on A and j elements are the smallest in some positive circuit of $-_A M$. The parameters i and j for a given oriented matroid $-_A M$ are called its *dual-orientation* and *orientation activities*. This second formula contains several results of the literature on counting acyclic orientations in graphs, regions in arrangements of hyperplanes and pseudohyperplanes, vertices of zonotopes, acyclic reorientations of oriented matroids [3,14–17,20,23,24] (details will be given in the second paper of the series).

Comparing these two expressions for $t(M; x, y)$, we get the *orientation/basis activity relations*

$$o_{i,j} = 2^{i+j} b_{i,j}$$

for all $i, j = 0, 1, \dots$.

The question arises of a bijective interpretation of these formulas [18]. The problem is to define a mapping from reorientations to bases compatible with the orientation/basis activity relations, i.e. an activity-preserving mapping. More precisely, the desired mapping should associate an (i, j)-active basis with an (i, j)-active reorientation, in such a way that each basis of M is the image of exactly 2^{i+j} (i, j)-active reorientations.

The active mapping, subject of the present paper and of three others in the series [11,10,12], provides an answer to the above question. Furthermore, it turns out that the active mapping not only preserves activities with the right multiplicities, as desired, but actually also preserves active elements. Moreover it preserves certain fundamental partitions of the ground set associated with active elements – the *active partitions* – to be defined in [11]. Reorienting arbitrary parts of the active partition of an (i, j)-active reorientation defines its *activity class*, having 2^{i+j} elements. The active mapping induces an *active bijection* between activity classes of reorientations and bases, which depends only on the reorientation class of the ordered oriented matroid. It can be also refined into an *active bijection* between reorientations and all subsets of elements. In particular, the active bijection specializes to a bijection between acyclic reorientations (regions) and subsets of elements containing no broken circuit [11,9,13]. Other specializations involve for instance acyclic orientations with unique sink in graphs [8], permutations and signed permutations [9]. As will be shown in Section 4 below and in further papers of the series, the active mapping behaves simply with respect to matroid duality.

Several constructions of the active mapping will be given in the series of papers. The first construction reduces the problem to the uniactive internal case, when $i = 1$ and $j = 0$. The uniactive internal case – the bounded case from a geometrical point of view – deals with bounded regions in real hyperplane arrangements and oriented matroids, with bipolar orientations in the case of graphs, which are mapped onto uniactive internal bases. The bounded case is the object of the present paper. Given a uniactive internal basis, a combinatorial algorithm computes a bounded region having this basis as fully optimal basis. This algorithm is simple, which is not so in the reverse direction. Then we prove that this construction is bijective. The decomposition of activities, and its application to construct the general active mapping, is dealt with in the second paper of the series [11]. In the third paper [10], we will provide a direct construction of the fully optimal basis of a bounded region by means of (pseudo)linear programming. Another construction of the active mapping uses deletion/contraction relations (as often in the context of Tutte polynomials). Finally, universal properties can be used to define the active mapping, particularly in the real case, and in the slightly more general case of Euclidean matroids. Inductive properties and universality will be presented in the fourth paper [12]. We emphasize that the active mapping is not invariant with respect to the ordering, despite the invariance of the number of bases with given activities.

Particular cases of the active mapping have already been published by the authors. These papers, which can be used to illustrate the general theory of the present series, deal with uniform and rank-3 oriented matroids [7], with graphs [8], and with supersolvable hyperplane arrangements [9]. Definitions and results of the present paper and of [11,12] are the subject of the thesis of the first author [6]. A partial solution to the problem of building a mapping compatible with the orientation/basis activity relations, different from the present one, is presented in [19] in the case of graphs, and, more generally, of regular matroids. A short extended abstract covering the whole series on the general active bijection has been published in [13].

The present paper deals with the uniactive cases $o_{1,0} = 2b_{1,0}$ and $o_{0,1} = 2b_{0,1}$ of the orientation/basis activity relations. These two cases being equivalent by matroid duality, it suffices to consider the *uniactive internal* case $o_{1,0} = 2b_{1,0}$. T. Zaslavsky has proven that the number of bounded regions of a real hyperplane arrangement is equal to the β-invariant of its matroid [24], a result generalized to oriented matroids by M. Las Vergnas in [16]. Suppose the set of hyperplanes is linearly ordered, with the hyperplane at infinity chosen as the smallest element. Then a bounded region corresponds to a (1, 0)-active reorientation, and β is the number of (1, 0)-active bases. Hence, up to factor 2 due to negating signs, the Zaslavsky–Las Vergnas result is the (1, 0) orientation/basis activity relation.

We construct in what follows a natural bijection between bounded regions and uniactive internal bases. Our main tool is the notion of *fully optimal basis*. Fully optimal bases have a simple combinatorial definition and several geometrical interpretations, strengthening in particular optimal bases of linear programming. An optimal basis is associated with one flat of dimension 0 – a vertex – with a maximality property (solution to a linear program). A fully optimal basis is equivalent to a maximal flag of flats – one in each dimension – each with a maximality property (unique solution to a lexicographic multiobjective pseudolinear program, see [10]).

The main results of the present paper are: (1) a bounded region has a unique fully optimal basis, (2) fully optimal bases provide a bijection between bounded regions and uniactive internal bases, (3) simple duality properties of fully optimal bases.

2. Preliminaries

We assume an elementary knowledge of matroids and oriented matroids. Background on oriented matroids can be found in [1] (see also [2]). For the convenience of the reader, we recall here the main definitions and technical results needed in what follows.

The content of Section 2.1 suffices for a combinatorial understanding of the paper. However, it turns out that the active mapping has enlightening geometrical interpretations in terms of topological oriented matroids (arrangements of signed pseudospheres) and also in terms of pseudolinear programming. Brief accounts of prerequisites are given in Sections 2.2 and 2.3.

2.1. Matroids and oriented matroids

We use the definition of oriented matroids in terms of circuits, i.e. a collection of signed sets satisfying certain properties — in particular, the *signed elimination property* [1, Sect 3.2].

In graphs, circuits resp. cocircuits are edge-sets of inclusion minimal cycles resp. cocycles signed accordingly with edge directions. Given a cycle resp. cocycle direction, an edge is signed $+$ if directed in the given direction, and $-$otherwise. Bases are edge-sets of spanning forests.

In central arrangements of signed real hyperplanes, defined by linear forms, the oriented matroid circuits are the sign-vectors of minimal linear dependencies of the linear forms.

Two signed sets C and D are *orthogonal* if we have $(C^+ \cap D^+) \cup (C^- \cap D^-) \neq \emptyset$ if and only if $(C^+ \cap D^-) \cup (C^- \cap D^+) \neq \emptyset$. By forgetting signs in an oriented matroid, we get its *underlying* (ordinary) matroid. Given an oriented matroid M, there is a unique oriented matroid M^*, the *dual* of M, whose underlying matroid is the dual of the matroid underlying M, such that the circuits of M and M^* are orthogonal [1, Sect. 3.4]. The circuits of M^* are called the *cocircuits* of M.

The *composition* of two signed sets C and D is the signed set $C \circ D$ defined by $(C \circ D)^+ = C^+ \cup (D^+ \setminus C^-)$ and $(C \circ D)^- = C^- \cup (D^- \setminus C^+)$. A composition $C \circ D$ is *conformal* if C and D have equal signs on their intersection. In an oriented matroid, a composition of circuits can be rewritten as a conformal composition of circuits [1, Prop.3.7.2].

Let B be a basis of a matroid M with rank $r = r(M)$. For $e \notin B$, the *fundamental circuit* of e with respect to B, denoted by $C_M(B; e) = C(B; e)$, is the unique circuit contained in $B \cup \{e\}$. For $b \in B$, the *fundamental cocircuit* of b with respect to B, denoted by $C^*_M(B; b) = C^*(B; b)$, is the unique cocircuit contained in $(E \setminus B) \cup \{b\}$. Alternately, $C^*(B; b)$ is the complement of the hyperplane – flat of rank $r(M) - 1$ – generated by $B \setminus \{b\}$. By orthogonality, we have $b \in C(B; e)$ if and only if $e \in C^*(B; b)$. In an oriented matroid, there are two opposite signed circuits resp. cocircuits with the same underlying circuit resp. cocircuit. We denote by $C(B; e)$ resp. $C^*(B; b)$ the signed fundamental circuit resp. cocircuit such that e resp. b is signed $+$. By (oriented) matroid duality, if B is a basis of M, then $E \setminus B$ is a basis of M^*, and for $b \in B$ we have $C^*_M(B; b) = C_{M^*}(E \setminus B; b)$.

Let M be an oriented matroid on $\{e_1, e_2, \dots, e_n\}$. We define the *(fundamental) tableau* of B in M as the $n\times n$ matrix with coefficients in $\{+, -, 0\}$, whose i-th column is the sign-vector of $C^*(B; e_i)$ if $e_i \in B$ and i-th row is the sign-vector of $-C(B; e_i)$ if $e_i \notin B$, and with 0 everywhere else. The consistency of this definition follows from the orthogonality property from oriented matroid duality. This definition of a tableau in an oriented matroid differs slightly from that in [1, Chap.10]. The present definition is more convenient for our purpose. Clearly, the fundamental tableau of $E \setminus B$ in M^* is the opposite of the transpose of the complete fundamental tableau of B in M.

A linear ordering of a set E is given by a indexation of its elements, $E = \{e_1, e_2, \dots, e_n\}_<$. A matroid is said to be *ordered* if its set of elements is linearly ordered. The *minimal basis* of an ordered matroid M on E is the unique basis $B_{\min} = \{f_0, f_1, \dots, f_{r-1}\}_<$ minimal for the lexicographic ordering of r-subsets induced by the linear ordering of E. The minimal basis is easily built by means of the Greedy Algorithm. For $0 \le i \le r - 1$, f_i is the smallest element not belonging to the closure of $\{f_0, f_1, \dots, f_{i-1}\}$. In particular, f_0 is the smallest non-loop element of E, and f_1 is the smallest non-loop element, different from f_0 and not parallel to it. As easily shown, any $b \in B_{\min}$ is the smallest in its fundamental cocircuit $C^*(B_{\min}; b)$, and, conversely, the smallest element of any cocircuit belongs to the minimal basis. Equivalently, the minimal basis is the set of all smallest elements of cocircuits of the matroid.

General activities will not be used in this paper. It suffices to define here a $(1, 0)$-active basis, or uniactive internal basis. A basis B of M is *internal*, or, equivalently, has external activity 0, if no element $e \in E \setminus B$ is the smallest in its fundamental circuit $C(B; e)$. Clearly, the minimal basis $B_{\min}$ is internal. An internal basis is *uniactive* if no basis element except e_1 is the smallest in its fundamental cocircuit. The property for a basis to be uniactive internal can easily be read off from its (unsigned) tableau.

A $(1, 0)$ orientation active oriented matroid, or bounded acyclic oriented matroid, is defined similarly. An oriented matroid is *acyclic*, or has *orientation activity* 0, if it contains no positive circuit. By the Farkás Lemma for oriented matroids [1, Cor. 3.4.6], every element belongs either to a positive circuit or to a positive cocircuit. Equivalently, an oriented matroid is acyclic if every element belongs to some positive cocircuit. An acyclic oriented matroid is *bounded* (with respect to e_1), or has *dual-orientation activity* 1, if all its positive cocircuits contain e_1. In the case of graphs (see [8]), an acyclic directed graph is *bounded* with respect to e_1 if and only if it is *bipolar* with respect to e_1, that is the extremities of e_1 are the unique source and unique sink of the directed graph.

The *reorientation* of an oriented matroid M on a subset of elements $A \subseteq E$, denoted by $-_AM$, is a resigning of the circuits of M defined by reversing signs of elements in A [1, Sect. 3.1]. We will usually make a slight abuse of language by identifying a reorientation $-_AM$ of M with its defining subset A. For instance, we will say that a reorientation $A \subseteq E$ is acyclic if the oriented matroid $-_AM$ is acyclic. Hence, there are always $2^{|E|}$ reorientations of M.

2.2. Geometrical representation

Oriented matroids can be topologically represented by arrangements of signed pseudospheres. In examples, we will use the equivalent, smaller by roughly a half, representation by spherical diagrams.

Let M be an oriented matroid (without loops) of rank $r = d + 1$ with elements $e_1, e_2, \dots, e_n$. In an arrangement of signed pseudospheres in S^d, a signed pseudosphere is an image e by a homeomorphism of S^d of the unit sphere S^{d-1}, with signs $+, -$ assigned to the two connected components of $S^d \setminus e$. In particular, a central hyperplane arrangement is transformed into a (pseudo)sphere arrangement by taking intersections with S^d. Given an oriented matroid of rank

$r = d + 1$, there is a arrangement of $(d - 1)$-dimensional signed pseudospheres in the sphere S^d such that the faces they determine are in a canonical 1–1 correspondence with the signed covectors of M. Here, we will only use rank-3 examples, equivalent to well-known pseudoline arrangements. For a background on general pseudosphere arrangements, see [1, def. 5.1.3]. It suffices to know that the mentioned 1–1 correspondence is obtained by associating with every point $x \in S^d$ the sign-vector $\sigma_x \in \{+, -, 0\}^n$ whose i-th component is $+$ resp. $-$, 0 if $x \in e_i^+$ resp. $x \in e_i^-$, $x \in e_i$.

We observe that up to a homeomorphism of S^d, we may suppose that $e_1 = S^{d-1}$ and $e_1 \cup e_1^+ = B^d$. Restricting the arrangement to B^d, we obtain a *spherical diagram* of M. We assume that no element of M is parallel to e_1. Then $e_2, \ldots, e_n$ constitute a pseudohyperplane arrangement restricted to the ball B^d, having all its vertices in B^d. The sphere e_1 is called the *hyperplane at infinity* of the diagram. A region is *bounded* if has no vertex in the hyperplane at infinity e_1. We also say e_1-*bounded* for short. Points not on e_1 are *at finite distance*. The following properties hold.

- Sign-vectors of points in $B^d \setminus \bigcup_{1 \le i \le n} e_i$ establish a bijection between regions of the diagram, i.e. connected components of $B^d \setminus \bigcup_{1 \le i \le n} e_i$, and maximal covectors of M with first component $+$.
- The mapping $x \mapsto \sigma_x^-$ is a bijection from regions to acyclic reorientations of M contained in $\{e_2, \ldots, e_n\}$. In particular, the oriented matroid M is acyclic if and only if all signs of some region, necessarily unique, are $+$. This region is called the *fundamental* region.
- A dimension-0 intersection of e_i's consists of either 1 point in e_1^+ or 2 points in e_1. These points – called *vertices* of the diagram – correspond to cocircuits of M. Spherical diagrams are different from projective representations of oriented matroids: opposite points on $e_1 = S^d$ are not identified. A cocircuit not containing e_1 is associated with two vertices at infinity, one for each opposite signed cocircuit. A cocircuit containing e_1 is associated with a unique vertex at finite distance.
- A dimension-1 intersection of e_i's is either a pseudocircle contained in e_1 or a pseudosegment joining two opposite vertices of e_1 with interior not in e_1. In both cases, we call it improperly a *pseudoline* of the diagram. Two vertices adjacent on a pseudoline are *conformal*: their common non-zero signs are equal. A part of a pseudoline defined by two adjacent vertices is called an *edge* of the diagram.
- Let C, D be two covectors, corresponding respectively to faces v and w. Then the *composition* $C \circ D$ corresponds to the face of the flat spanned by v and w, which contains v and is in the side of w of all pseudohyperplanes containing v. In particular, if C and D are two conformal cocircuits, then $C \circ D = D \circ C$ corresponds to the edge joining the corresponding vertices.

Arrangements of affine real hyperplanes, with an added hyperplane at infinity, constitute the real case of spherical diagrams. For d $=$ 2 spherical diagrams of oriented matroids are circular diagrams of pseudoline arrangements (see [1, Chap. 6]). These two examples provide an intuition for the geometrical interpretation of the constructions in this paper, as well as for the extension of linear programming to oriented matroids, or *pseudolinear programming* (see [1, Chap. 10]).

An example of circular diagram is shown in Fig. 1. The gray region is the fundamental region, corresponding to the positive maximal covector. It is 1-bounded. The associated cocircuit is shown for each vertex of the diagram. For instance, let us consider the cocircuit C^*, associated with the vertex v. Since v is on 127, these elements do not occur in C^*. On the other hand, v is not on 3456, hence these elements occur in C^*. The vertex v and the fundamental region are on the same side of pseudoline 3, hence 3 is positive in C^*, we simply write 3. The vertex v and the fundamental region are separated by pseudoline 4, hence 4 is negative in C^*, we write $\bar{4}$. Continuing, we eventually get $C^* = 3\bar{4}\bar{5}6$.

2.3. Pseudolinear programming

A *linear program* in R^d is defined by a polytopal region — the *feasible region*, intersection of closed half-spaces defined by affine hyperplanes — the *program hyperplanes*, and by a linear form on R^d — the *objective function*. Basic property: the objective function always attains a maximum on a non-empty and bounded feasible region.

For a combinatorial version of this theorem, let us represent the objective function by directing in its increasing direction all edges defined by the hyperplanes of the program not parallel to it. Then a

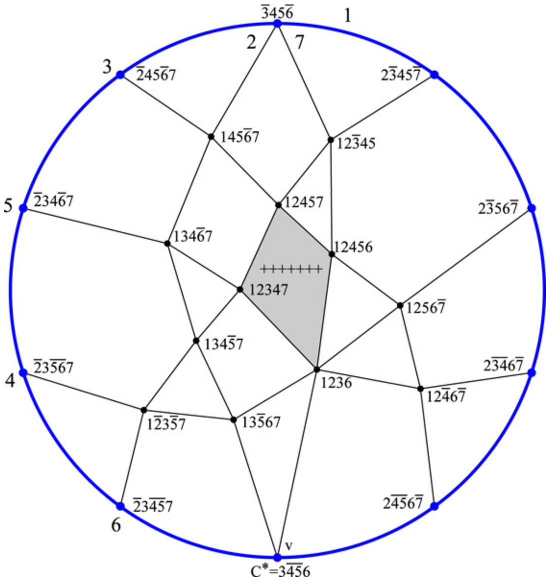


Fig. 1. Cocircuits in a circular diagram of pseudolines.

key observation is that, in the real case, a vertex v of the feasible region is a maximum of the objective function if and only if *no edge of the feasible region incident to v is outgoing from v*.

In the paper, an *oriented matroid program* $(M; p, f)$, or *pseudolinear program*, is defined by an acyclic oriented matroid on $E = \{e_1, e_2, \ldots\}$ with hyperplane at infinity $p = e_1$, objective function $f \in E$, and such that the feasible region is the fundamental region. In particular, the feasible region is always on the positive side of f. No loss in generality results from these conventions, slightly different from [1].

We construct the *program graph* by directing the edges of the feasible region not parallel[3] to the objective function f in the direction going from the negative side of f towards its positive side. This can be done combinatorially, i.e. by means of sign-vectors. Note that in the oriented matroid case, the program graph may contain directed cycles (this cannot happen in the real case). Nevertheless, the main theorem remains valid.

Theorem 2.1 (*[1, Th. 10.1.13]*). *The graph of a pseudolinear program on a bounded feasible region contains at least one vertex with no outgoing edge.*

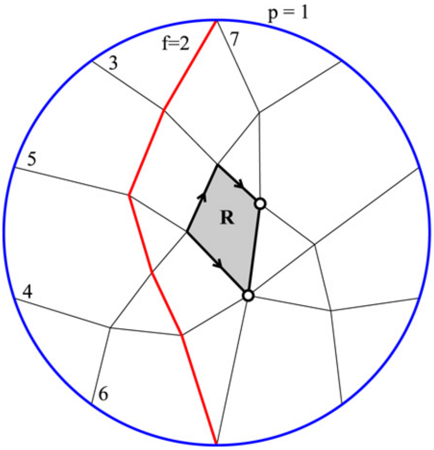
Any vertex of the program graph with no outgoing directed edge is a solution to the pseudolinear program. We say here that such a vertex is an *optimal vertex*. In the example of Fig. 2, there are two optimal vertices.

The main definition of the present paper, to be introduced in the next section, is a refinement of the classical notion of optimal basis, used in the Simplex Criterion to characterize optimal vertices.

We recall that a basis B of a (pseudo)linear program of dimension d, i.e. $r = d + 1$ independent hyperplanes, is said to be *optimal* if (i) $p \in B$, (ii) $f \notin B$, (iii) the fundamental cocircuit $C^*(B; p)$ is positive, (iv) the fundamental circuit $C(B; f)$ has p as its unique negative element. This definition is the acyclic case of the definition in [1, Cor. 10.2.8].

---

3 Two pseudohyperplanes, or a pseudohyperplane and a pseudoline, are *parallel* if their intersection is contained in the hyperplane at infinity.



137

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3			(+)				
4	+		−	−			+
5	+		−		−		+
6	+		+			−	−
7							+

145

	1	2	3	4	5	6	7
1	+						
2	+	−		−	(+)		
3	+		−	+	−		
4				+			
5					+		
6	+			−	+	−	
7				+	−		−

147

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			−		−	−
7							+

157

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				(−)	+		+
5					+		
6	+				−	−	−
7							+

Fig. 2. Optimal bases.

Proposition 2.2 (*The Simplex Criterion [1, Cor. 10.2.8]*). *A vertex v of the feasible region is an optimal vertex of a (pseudo)linear program if and only if there is an optimal basis $\{b_1 = p, b_2, \ldots, b_r\}$ such that $v = b_2 \cap b_3 \cap \cdots \cap b_r$.*

Properties of bases in oriented matroid programming are conveniently described by their fundamental tableaux, analogue to the tableaux in linear programming.

We say that B is *supported* by the vertex $v = b_2 \cap b_3 \cap \cdots \cap b_r$. The condition $C^*(B; p)$ positive expresses that v is a vertex of the feasible region. Geometrically, the condition $C^-(B; f) = \{p\}$ expresses that p does not meet the interior of the region $f^+ \cap b_2^+ \cap b_3^+ \cap \cdots \cap b_n^+$. In the real case, the condition $C^-(B; f) = \{p\}$ means that the cone $b_2^+ \cap b_3^+ \cap \cdots \cap b_n^+$ – whose closure contains the feasible region – is in the negative closed half-space defined by the hyperplane parallel to f through v, implying that v is optimal.

Optimal bases are generally not unique.

In the example of Fig. 2 there are 4 bases supported by optimal vertices, namely 137 145 147 157. The 3 bases 137 147 157 are optimal. The basis 145, associated with the optimal vertex $e_4 \cap e_5$, is not optimal. The circled sign in row 2 column 5 of its tableau should be a $-$, however it is actually $+$, since the vertex $e_1 \cap e_4 \cap e_5^+$ is in e_2^+. The circled signs in tableaux of bases 137 and 157 will be explained in the next section. Another example is provided in Fig. 4.

3. Fully optimal bases

We introduce in this section the main new notion of the paper, namely that of *fully optimal basis* in a bounded acyclic ordered oriented matroid. This definition will be extended to general oriented matroids in [11], by means of duality and decompositions into bounded acyclic minors.

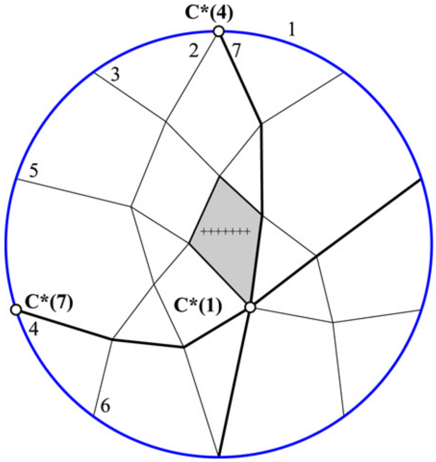
Definition 3.1. Let M be an oriented matroid on a linearly ordered set $E = \{e_1, e_2, \ldots\}_<$.

We say that a basis B of M is *fully optimal* if and only if

(i) for every $e \in E \setminus B$, the signs in $C(B; e)$ of e and $\min C(B; e)$ are opposite, and
(ii) for every $b \in B \setminus e_1$, the signs in $C^*(B; b)$ of b and $\min C^*(B; b)$ are opposite.

Equivalently, B is fully optimal if and only if

(i′) the first non-zero sign of each row of its tableau in M is $+$, and
(ii′) the first non-zero sign of each column except the first one is $-$.



B=147

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			−		−	−
7							+

Fig. 3. A fully optimal basis.

The properties (i) and (ii), or (i′) and (ii′), immediately imply that $f_0 = e_1 \in B$ and that $f_1 \notin B$, where f_1 is the first element of M not parallel to $f_0 = e_1$.

A fully optimal basis $B = \{b_1 = e_1, b_2, \ldots, b_r\}_<$ is in particular an optimal basis of the program $(M; e_1, f_1)$ defined by the hyperplane at infinity $p = e_1$, the objective function $f = f_1$, and with the fundamental region of M as feasible region. By the Simplex Criterion, the vertex $v = b_2 \cap b_3 \cap \cdots \cap b_r$ is an optimal vertex for this program.

For example, comparing Figs. 2 and 3, the basis 147 is both optimal and fully optimal, whereas bases 137 and 157 are optimal but not fully optimal, as shown by the circled signs in their tableaux.

We note some simple consequences of Definition 3.1.

Proposition 3.2. *Let M be an ordered oriented matroid on $\{e_1, e_2, \ldots\}_<$.*

(i) *If M has a fully optimal basis then it is acyclic, and its fundamental region is bounded with respect to the hyperplane at infinity e_1.*
(ii) *A fully optimal basis is internal and uniactive.*

Proof. Let B be a fully optimal basis of M.

(i) It follows immediately from the row property in Definition 3.1 that the composition $C^*_M(B; e_1) \circ C^*_M(B; e_2) \circ \cdots \circ C^*_M(B; e_r)$ is a positive covector supported by the entire set of elements E. Hence M is acyclic.

We prove that the fundamental region is bounded. Let D be a positive cocircuit of M. Then $D \setminus B \neq \emptyset$, otherwise the basis B is contained in the hyperplane $E \setminus D$. Let $e \in D \setminus B$. By the column property in Definition 3.1, if $e_1 \notin D$, then D has $+$ sign and $C_M(B; e)-$ sign on their non-empty intersection, contradicting the orthogonality property in oriented matroids. Hence $e_1 \in D$, thus the fundamental region has no vertex at infinity, i.e. is bounded.

(ii) If an element e not in B is the smallest in its fundamental circuit, then the smallest element of the row e is $-$, contradicting the row condition of Definition 4.1. Hence B is internal. If an element $e \in B$ is the smallest in its fundamental cocircuit, then the smallest element of the column e is $+$. Therefore, by the column condition of Definition 3.1, we have $e = e_1$. It follows that B is uniactive. □

In Fig. 2, the basis 137 is not uniactive, and 157 is not internal, whereas a fully optimal basis is uniactive internal by Proposition 3.2. Another example, with uniactive internal bases being optimal but not fully optimal is given by Fig. 4.

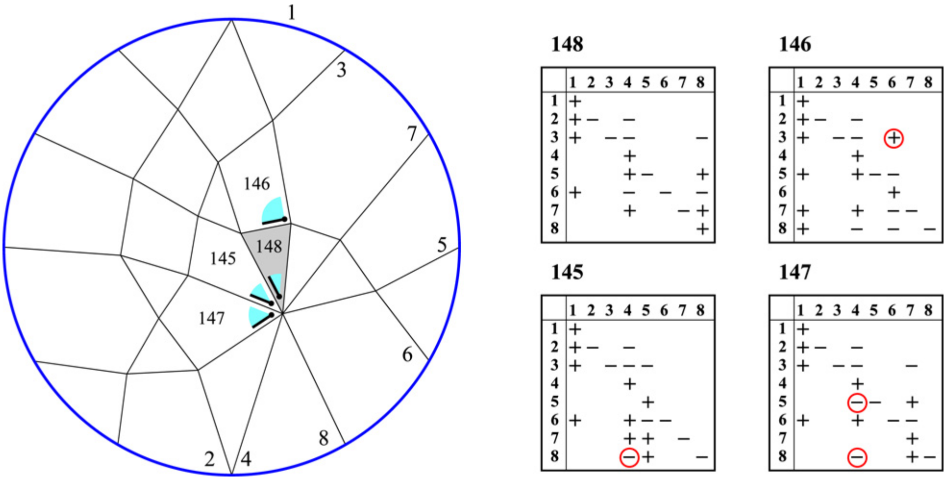


Fig. 4. The tableaux of 4 bases with respect to the gray region.

Proposition 3.3. *Let M be an oriented matroid on a linearly ordered set $E = \{e_1, e_2, \ldots, e_n\}_<$, and $B = \{b_1, b_2, \ldots, b_r\}_<$ be a uniactive internal basis. Set $E \setminus B = \{c_1, c_2, \ldots, c_{n-r}\}_<$.*

Then B is fully optimal if and only if

(i) *[Adjacency] the covector $C^*_M(B; b_1) \circ C^*_M(B; b_2) \circ \cdots \circ C^*_M(B; b_r)$ is positive, and,*

(ii) *[Dual-Adjacency] the vector $C_M(B; c_1) \circ C_M(B; c_2) \circ \cdots \circ C_M(B; c_{n-r})$ has $e_1 = b_1$ as its unique negative element.* □

The conditions internal or uniactive cannot be removed. In the graph on 3 vertices a, b, c with edges $1 = ab$ $2 = cb$ $3 = ac$ $4 = ac$ the basis 12 is internal, not uniactive, the basis 14 is not internal, hence none is fully optimal. However both satisfy the adjacency and dual-adjacency properties.

Proposition 3.3 has a simple geometrical interpretation. As recalled in Section 2.2, covectors are sign-vectors of points of a topological representation, i.e. are in 1–1 correspondence with the faces of an arrangement of signed pseudospheres representing the oriented matroid. If the point is a vertex of the arrangement – a zero-dimensional flat – then the covector is a cocircuit, and conversely. Otherwise the point is an interior point of a unique face of dimension ≥ 1 of the arrangement, associated with the corresponding covector. In a spherical diagram, we are concerned only with sign-vectors where e_1 is non-negative.

Given a basis $B = \{b_1, b_2, \ldots, b_r\}_<$, the cocircuit $C^*_M(B; b_1)$ is associated with the vertex $v = b_2 \cap b_3 \cap \cdots \cap b_r$. The cocircuits $C^*_M(B; b_1), C^*_M(B; b_2), \ldots, C^*_M(B; b_i)$ are vertices of the simplex with facets supported by $b_1, b_2, \ldots, b_r$ representing B. There are 2^{r-1} regions in a pseudosphere arrangement representing M associated with covectors of type $C^*_M(B; b_1) \circ \pm C^*_M(B; b_2) \circ \cdots \circ \pm C^*_M(B; b_r)$. They are on the positive side of b_1 and appear in the spherical diagram of M. For such a region, and for $i \geq 1$, let w_i be the $(i-1)$-dimensional face associated with the i-th covector $C^*_M(B; b_1) \circ \pm C^*_M(B; b_2) \circ \cdots \circ \pm C^*_M(B; b_i)$. We have $w_1 = v$, then w_i is supported by $b_{i+1} \cap \cdots \cap b_r$, $i = 1 \ldots r-1$, and w_r is the considered region among the above 2^{r-1} regions. The sequence $w_1, w_2, \ldots, w_r$ is a sequence of nested faces – or flag – of this region, which is maximal in the sense that it contains one face in each dimension. We say that regions of this type are *adjacent* to B.

Now assume that B is a fully optimal basis of M. Then, by the adjacency property of Proposition 3.3, the fundamental region of M corresponds to the positive covector $C^*_M(B; b_1) \circ C^*_M(B; b_2) \circ \cdots \circ C^*_M(B; b_r)$ and is adjacent to B. Moreover, by the dual-adjacency property of Proposition 3.3, $-_{e_1}M^*$ has a positive covector, corresponding to a region of a spherical diagram representing M^*, adjacent to the basis $E \setminus B$ of M^*. So, B is fully optimal in M if and only if both M corresponds to one of the 2^{r-1} regions adjacent to B and $-_{e_1}M^*$ corresponds to one of the 2^{n-r-1} regions adjacent to $E \setminus B$. This geometric property, involving a representation of M and of its dual, characterizes fully optimal bases.

See also Proposition 4.2, where we shall see that this property for a uniactive internal basis determines one and only one bounded region. See also Section 5 for further interpretation of this duality property.

A complete interpretation of fully optimal bases in terms of linear programming will be given in [10]. As mentioned before, a fully optimal basis $B = \{b_1, b_2, \ldots, b_r\}_<$ is an optimal basis of the program $(M; p = e_1, f = f_1)$, and the vertex $v = b_2 \cap b_3 \cap \cdots \cap b_r$ is an optimal vertex for this program. However, the vertex v satisfies stronger extremal properties. The vertex v is the unique solution to a lexicographic multiobjective pseudolinear program, lexicographic multiprogram for short, defined by the minimal basis $B_{\min}$. Furthermore, not only v, but also each of the flats $b_i \cap b_{i+1} \cap \cdots \cap b_r$ can be characterized by certain extremal properties, expressible in terms of linear programming. The face w_i is the unique solution to a lexicographic multiprogram among all $(i-1)$-dimensional faces of the fundamental region containing w_{i-1} (this condition is void for $i = 1$). The reformulation of Definition 3.1 given by Proposition 3.3 can be considered as a strengthening of the Simplex Criterion.

Fig. 4 shows the tableaux of 4 internal uniactive bases with respect to the bounded region R, shaded in gray. The basis 148 is a fully optimal basis of R. The bases 145, 146, and 147 are not fully optimal as the circled signs in their tableaux should be opposite. The 4 bases are optimal bases of the program $(M; 1, 2)$ with fundamental region R. If 8 resp. 6, 58 is reoriented, then the tableau of 145 resp. 146, 148 becomes the tableau of a fully optimal basis in the region $R' = -_8R$ resp. $-_6R$, $-_{58}R$. For each region, we have represented the nested faces $v = w_1 \subset w_2 \subset w_3 = R'$ corresponding to the covectors $C^*_M(B; b_1) \circ C^*_M(B; b_2) \circ \cdots \circ C^*_M(B; b_i)$ for $i = 1, 2, 3$ signed with respect to the fundamental region R'.

We end this section by two other reformulations of Definition 3.1.

Proposition 3.4. *Let M be an oriented matroid on a linearly ordered set $E = \{e_1, e_2, \ldots, e_n\}_<$, and $B = \{b_1, b_2, \ldots, b_r\}_<$ be a basis of M. Set $E \setminus B = \{c_1, c_2, \ldots, c_{n-r}\}_<$. The following properties* (i)–(iii) *are equivalent.*

(i) *B is fully optimal.*
(ii) *B is internal, $C^*_M(B; b_1)$ is positive, and for $2 \le i \le r$ the smallest element of $C^*_M(B; b_i)$ is negative and $C^*_M(B; b_i) \setminus \bigcup_{1 \le j < i} C^*_M(B; b_j)$ is positive.*
(iii) *B is uniactive, for $1 \le i \le n - r$ the smallest element of $C_M(B; c_i)$ is negative and $C_M(B; c_i) \setminus \bigcup_{1 \le j < i} C_M(B; c_j)$ is positive.* □

The example given for Proposition 3.3 shows that the condition internal resp. uniactive cannot be removed from (i) resp. (ii).

4. The active bijection

In this section, we establish our main result: a bounded region admits exactly one fully optimal basis, providing a bijection between bounded regions and uniactive internal bases. Our proof is indirect. First, given a uniactive internal basis, we construct by means of a simple algorithm the unique bounded region for which this basis if fully optimal. Then we prove that a bounded region admits at most one fully optimal basis. It follows that the mapping from uniactive internal bases to bounded regions on the positive side of infinity given by the algorithm is an injection. As well-known, the number of bounded regions on one side of infinity is equal to the number of uniactive internal bases [24,16], hence this injection is actually a bijection, whose converse is called the *active mapping* or *active bijection*. The existence and unicity follow.

A direct construction of the fully optimal basis of a given bounded region, by means of (pseudo)linear programming, will be the object of the forthcoming paper [10]. A simple construction of the fully optimal basis by deletion/contraction will be also given in the forthcoming paper [12].

A more complete statement of the following lemma – with converse and dual – is given in [8] Prop. 3.2 for graphs. It generalizes to matroids without change. We give a proof of Lemma 4.1 for completeness.

Lemma 4.1. *Let M be a matroid on a linearly ordered set E, and $B = \{b_1, b_2, \dots, b_r\}_<$ be an internal basis of M.*

Then b_i is the smallest element of $E \setminus \bigcup_{1\le j<i} C^*(B; b_j)$ for $i = 1, 2, \dots, r$.

Proof. Let $e = \min \left(E \setminus \bigcup_{1\le j<i} C^*(B; b_j)\right)$. Suppose $e < b_i$. We have $B \cap \left(E \setminus \bigcup_{1\le j<i} C^*(B; b_j)\right) \subseteq \{b_i, b_{i+1}, \dots, b_r\}$, hence $e \notin B$. Set $C = C(B; e)$. For $1 \le j < i$ if $b_j \in C$, we have $e \in C^*(B; b_j)$, hence $C \cap \{b_1, \dots, b_{i-1}\} = \emptyset$. It follows that $C \cap B \subseteq \{b_i, \dots, b_r\}$, hence e is the smallest in C, contradicting the hypothesis that B is internal. Therefore $e \ge b_i$, and we have $e = b_i$. □

Proposition 4.2. *Let M be an oriented matroid on a linearly ordered set E with smallest element e_1, and B be a uniactive internal basis of M.*

There exists a unique reorientation $A \subseteq E \setminus \{e_1\}$ such that B is fully optimal in $-_A M$. The subset A can be constructed by either of the following three algorithms.

Moreover, the (bounded acyclic) oriented matroid $-_A M$ is invariant under reorientation of M. In other words, there is a uniquely defined region of the topological representation of M associated with B.

Algorithm 1. Let $B = \{b_1 = e_1, b_2, \dots, b_r\}_<$.

Set $A_1 = (C^*(B; b_1))^-$.

For $2 \le i \le r$, set

$$A_i = A_{i-1} + \left(\left(C^*(B; b_i)\right)^{\epsilon_i} \setminus \bigcup_{1\le j<i} C^*(B; b_j) \right)$$

where ϵ_i is the sign of $a = \min C^*(B; b_i)$ if $a \notin A_{i-1}$, minus this sign otherwise.

Then $A = A_r$.

Algorithm 2. Let $E \setminus B = \{c_1 = e_2, c_2, \dots, c_{n-r}\}_<$.

Set $A_1 = (C(B; c_1) \setminus \{e_1\})^{\epsilon_1}$, where ϵ_1 is the sign of the smallest element e_1 of $C(B; c_1)$.

For $2 \le i \le n - r$, set

$$A_i = A_{i-1} + \left((C(B; c_i))^{\epsilon_i} \setminus \bigcup_{1\le j<i} C(B; c_j) \right)$$

where ϵ_i is the sign of $a = \min C(B; c_i)$ if $a \notin A_{i-1}$, minus this sign otherwise.

Then $A = A_{n-r}$.

Algorithm 3. Let $E = \{e_1, e_2, \dots, e_n\}_<$.

Set $A_1 = \emptyset$.

For $2 \le i \le n$

– if $e_i \in B$, let $a = \min C^*(B; e_i)$, then, if $a \in C^*(B; e_i)^+ \setminus A_{i-1}$ or $a \in C^*(B; e_i)^- \cap A_{i-1}$, set $A_i = A_{i-1} + e_i$,

– if $e_i \notin B$, let $a = \min C(B; e_i)$, then, if $a \in C(B; e_i)^+ \setminus A_{i-1}$ or $a \in C(B; e_i)^- \cap A_{i-1}$, set $A_i = A_{i-1} + e_i$.

Then $A = A_n$.

Proof. We prove the validity of Algorithm 1. Clearly, $C^*(B; b_1)$ is positive in $-_{A_1} M$, hence in $-_A M$ since $(A \setminus A_1) \cap C^*(B; b_1) = \emptyset$. Let $2 \le i \le r$. Since B is uniactive, the smallest element a_i of $C^*(B; b_i)$ is $< b_i$, hence is in $E \setminus B$. Necessarily a_i is in $\bigcup_{1\le j<i} C^*(B; b_j)$, otherwise $C(B; a_i)$ would not contain any b_j with $j < i$, hence a_i would be the smallest in $C(B; a_i)$ contradicting B internal. Hence all elements of $C^*_{-_{A_i} M}(B; b_i) \setminus \bigcup_{1\le j<i} C^*(B; b_j)$ have the same sign, opposite to the sign of a_i. The same property holds in $-_A M$ since $A_i \subseteq \bigcup_{1\le j\le i} C^*(B; b_j)$, and by definition of the A_i's we have $(A \setminus A_i) \cap \bigcup_{1\le j\le i} C^*(B; b_j) = \emptyset$. Thus B satisfies (ii) of Lemma 5.1 in $-_A M$, hence is fully optimal.

This proof shows that if A is such that B is fully optimal in $-_A M$, then necessarily we have $A \cap (C^*(B; b_i)) \setminus \bigcup_{1\le j<i} C^*(B; b_j) = A_i \setminus A_{i-1}$ for $i = 1, 2, \dots, r$. Therefore A is unique.

The validity of Algorithm 2 can be proved similarly, by matroid duality. Algorithm 3 is a reformulation of Algorithms 1 and 2.

The invariance of $-_A M$ follows from the observation that, starting from $M' = -_S M$, we get $A' = A \Delta S$. Hence $-_{A'} M' = -_{A\Delta S}(-_S M) = -_{(A\Delta S)\Delta S} M = -_A M$. □

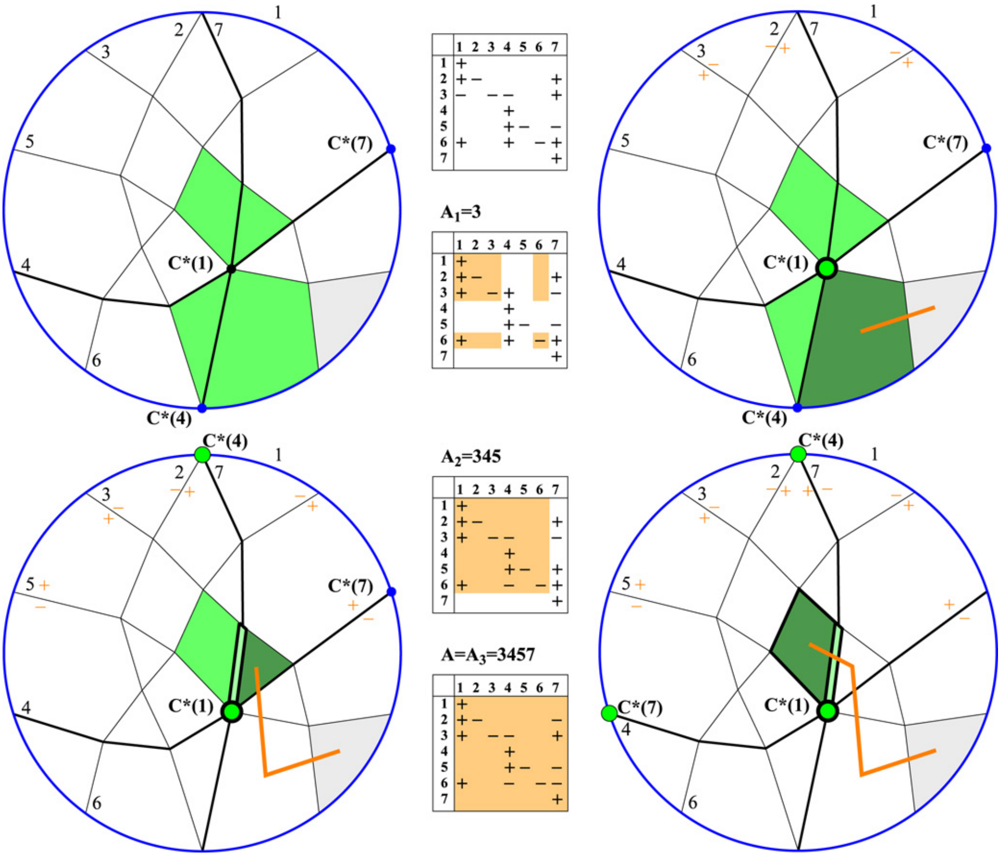


Fig. 5. Algorithm 1 for basis 147.

We point out that the algorithms computing the reorientation A in Proposition 4.2 depend only on the tableau of B. Algorithm 1 consists in reorienting some elements of successive columns, so that all elements that have not been considered yet are signed $+$ in the same column of the final tableau while the smallest element of this column is signed $-$. Similarly and dually, Algorithm 2 consists in considering successive rows, whereas Algorithm 3 consists in reorienting directly successive elements if necessary.

Geometrically, the region given by the final reorientation does not depend on the initial fundamental region. The first algorithm in Proposition 4.2 consists in finding the region associated with the basis by fixing step by step faces $w_1 \subset w_2 \subset \cdots \subset w_r$ of this region. It amounts to deleting at each step half of the 2^r regions adjacent to B on both sides of e_1, cf. Proposition 3.3. At the first step, the orientations of the elements of $C^*(B; b_1)$ are fixed, so that the final region is on the positive side of these elements. Then at each step, one of the two vertices representing the cocircuits $\pm C^*(B; b_i)$ is chosen to be on the side of b_i containing the final region. Namely the vertex on the negative side of the pseudohyperplane *min* $C^*(B; b_i)$, whose orientation has been fixed at a previous step.

The 4 diagrams of Fig. 5 show the 4 reorientations $-_{A_i}M$ for $i = 0, 1, 2, 3$, with $A_0 = \emptyset$, associated with the 3 steps of Algorithm 1 in rank $r = 3$. The fundamental region in the first diagram is shaded in light gray. The regions shaded in gray in the first picture are the $2^r = 8$ regions associated with the maximal covectors $\pm C^*(B; b_1) \circ \pm C^*(B; b_2) \circ \pm \ldots \circ \pm C^*(B; b_n)$. Only 4 are shown in Fig. 5, those in e_1^+. Each step of the algorithm divides the number of gray regions by 2.

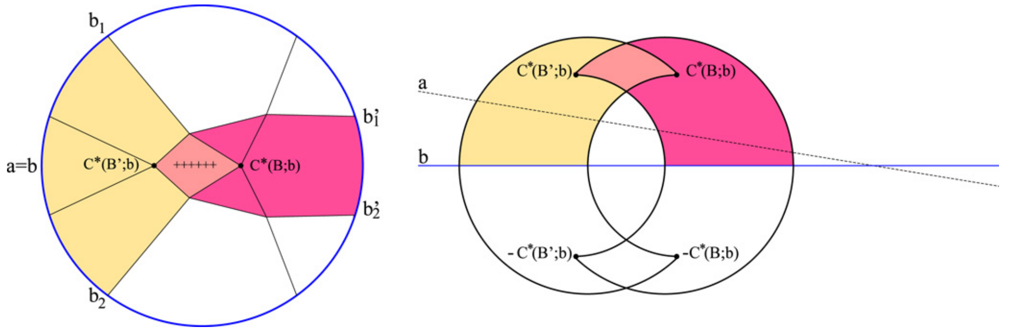


Fig. 6. Possible figure and impossible figure (The Crescent Lemma).

Proposition 4.3. *An ordered oriented matroid admits at most one fully optimal basis.*

The key to the proof of Proposition 4.3 is the following lemma.

Lemma 4.4 (*The Crescent Lemma*). *Let M be an oriented matroid on a set E, and B, B' be two distinct bases of M. Let $a \in E \setminus (B \cup B')$ and $b \in B \cap B'$. Set $D = C^*(B; b)$, $D' = C^*(B'; b)$, $C = C(B; a)$, and $C' = C(B'; a)$. Suppose that*

(i) $B' \cap D \subseteq D^+$,
(ii) $B \cap D' \subseteq D'^+$,
(iii) $(B' \cap D \cap C') - b + a \subseteq C'^+$, *and*
(iv) $(B \cap D' \cap C) - b + a \subseteq C^+$.

Then $(B \cap D' \cap C) - b = \emptyset$ *and* $(B' \cap D \cap C') - b = \emptyset$.

The Crescent Lemma says in particular that, in a topological oriented matroid, if we have two simplices (half-crescents) B, B' with a common pseudohyperplane b such that the vertex not in b of each simplex lies inside the other one, then no pseudohyperplane a can meet the interiors of both simplices on the same side of b without crossing b at the frontier of the simplices. See Fig. 6. Precisely, this figure shows two bases $B = \{b, b_1, b_2\}$ and $B' = \{b, b'_1, b'_2\}$, satisfying the above conditions. The vertex $b_1 \cap b_2$, resp. $b'_1 \cap b'_2$, corresponds to $C^*(B; b) = D$, resp. $C^*(B'; b) = D'$. Hypotheses (i) and (ii) of the lemma mean that the fundamental region corresponding to M is the region not touching b between these two vertices. Thereby, hypotheses (iii) and (iv) assume that an element $a \notin B \cup B'$ cuts both simplices defined by B with apex D, and B' with apex D' (but does not cut the fundamental region, nor the faces of the simplices contained in b). The conclusion of the lemma states that, under these conditions, a must be parallel to b, that is equal to b on the first figure. In other words, the second figure is impossible as a figure extracted from a pseudosphere arrangement.

Proof. Let D'' be a cocircuit obtained by elimination of b from $-D$ and D'. We have $B \cap D = \{b\}$, hence $B \cap (D \cup D') \subseteq (D' \setminus D) + b$. Therefore $B \cap D'' \subseteq D' \setminus D$. Since $B \cap D' \subseteq D'^+$ by (ii), it follows that $B \cup D'' \subseteq D''^+$. Similarly we have $B' \cap D'' \subseteq D''^-$.

Since $C - a \subseteq B$, we have $(C - a) \cap D'' \subseteq B \cap D'' \subseteq D''^+$. Similarly $(C' - a) \cap D'' \subseteq D''^-$.

Suppose $a \in D''$. Then by (iv) $C \cap D'' \subset (C - b) \cap D' + a \subset C^-$. Hence, by orthogonality, $(C - a) \cap D'' \subseteq D''^+$ implies $a \in D''^-$. Similarly, using C' and (iii), we get $a \in D''^+$. We have a contradiction, therefore $a \notin D''$. It follows, by orthogonality again, that $C \cap D'' = \emptyset$ and $C' \cap D'' = \emptyset$.

For all $e \in D \Delta D'$, let D''_e be a cocircuit containing e obtained by elimination of b from $-D$ and D'. We have $B \cap D' - b \subseteq D' \setminus D$ and $B' \cap D - b \subseteq D \setminus D'$. Hence $(B \cap D') \cup (B' \cap D) - b \subseteq \bigcup_{e \in D \Delta D'} D''_e$. It follows that C, contained in the complementary of $\bigcup_{e \in D \Delta D'} D''_e$, does not meet $B \cap D' - b$.

Similarly C' does not meet $B' \cap D - b$. $\square$

Proof of Proposition 4.3. Let M be an oriented matroid on a linearly ordered set E. Suppose for a contradiction that there are two distinct fully optimal bases $B = \{b_1, b_2, \ldots, b_r\}_<$ and $B' = \{b'_1, b'_2, \ldots, b'_r\}_<$.

By Lemma 4.1 there is a smallest integer $i \geq 1$ such that $C^*(B; b_i) \neq C^*(B'; b'_i)$. We have $b_j = b'_j$ for $1 \leq j \leq i$. Set $b = b_i = b'_i$, $D = C^*(B; b)$ and $D' = C^*(B'; b)$.

We have $B \cap D' - b \neq \emptyset$. Otherwise $B \cap D' = \{b\}$, therefore $D' = C^*(B; b)$, and thus $D = D'$, contradicting $D \neq D'$. Similarly $B' \cap D - b \neq \emptyset$.

(0) Let a be the smallest element of

$$\left(\bigcup_{e\in B\cap D'-b} C^*(B; e)\right) \cup \left(\bigcup_{e\in B'\cap D-b} C^*(B'; e)\right).$$

Set $C = C(B; a)$, $C' = C(B'; a)$.

(1) $a \notin B \cup B'$

By symmetry, we may suppose notation such that a is the smallest in $C^*(B; e)$ for some $e \in B \cap D' - b$.

(1.1) $a \notin B$

Otherwise we have $a = e$ since $a \in B \cap C^*(B; e) = \{e\}$. Hence by (0) a is the smallest in $C^*(B; a)$, implying $a = e_1$ since B is uniactive, by Proposition 4.2. We have $a = e = e_1 \in B' \cap D' = \{b\}$, hence $a = b$, which contradicts $a = e \in B \cap D' - b$.

(1.2) $a \notin B'$

For a contradiction, suppose $a \in B'$

(1.2.1) $a \notin D$

Otherwise, since $a \notin B$ and $b \in B$, we have $a \neq b$, hence $a \in B' \cap D - b$. Therefore, by (0) we have a smallest in $C^*(B'; a)$, and again $a = e_1$ since B' is uniactive. But $e_1 \in B$, contradicting (1.1).

(1.2.2) $a > b$

Otherwise, since $a \in B'$, we have $a \in \{b_1, b_2, \ldots, b_i\} \subseteq B$ contradicting (1.1).

(1.2.3) $a \notin D'$

Otherwise we have $a \in B' \cap D' = \{b\}$, hence $a = b$ contradicting (1.2.2).

(1.2.4) $C \cap D' \subseteq C^+ \cap D'^+$

Let $x \in C \cap D'$. We have $x \neq a$ by (1.2.3), hence $x \in C - a \subseteq B$. We have $x \neq b$, otherwise $b \in C$, hence $a \in D$, contradicting (1.2.1). Hence $x \in (C - a) \cap D' - b \subseteq B \cap D' - b$. Therefore by (0) a is the smallest element of $C^*(B; x)$. Since $x > a$, we have $x > e_1$. It follows from (ii) of Proposition 3.3 that $a \in (C^*(B; x))^-$. Hence by orthogonality $x \in C^+$.

We have $x \in B$ and $x > a > b$, hence $x \notin C^*(B; b_j) = C^*(B'; b'_j)$ for $1 \leq j < i$. Therefore $x \in D' \setminus \bigcup_{1\leq j<i} C^*(B'; b_j) \subseteq D'^+$ by (ii) of Proposition 4.3.

(1.2.5) $C \cap D' \neq \emptyset$

We have $a \in D = C^*(B; e)$, hence $e \in C = C(B; a)$. Since $e \in D'$, we have $e \in C \cap D'$.

(1.2.4) and (1.2.5) contradict the orthogonality property, hence (1.2) holds.

(2) $B' \cap D \subseteq D^+$

Let $b'_k \in D$, $k \neq i$. Suppose $b'_k \in C^*(B; b_j)$ for some $1 \leq j < i$. We have $b'_k \in C^*(B'; b'_j) = C^*(B; b_j)$, hence $b'_k = b'_j = b_j$. But $b_j \in D = C^*(B; b_i)$ implies $j = i$, contradicting $k \neq i$. It follows that $b'_k \in D \setminus \bigcup_{1\leq j<i} C^*(B; b_j) \subseteq D^+$ by (ii) of Proposition 4.3.

(3) $B \cap D' \subseteq D'^+$

Same proof as that of (2).

(4) $(B' \cap D \cap C') - b + a \subseteq C'^+$

We have $C' = C(B'; a)$, hence $a \in C'^+$. Let $b'_j \in D \cap C' - b$. Since $b'_j \in C(B'; a)$, we have $a \in C^*(B'; b'_j)$, hence a is the smallest in $C^*(B'; b'_j)$. By (ii) of Proposition 4.3 it follows that $a \in \left(C^*(B'; b'_j)\right)^-$, hence by orthogonality $b'_j \in \left(C(B'; a)\right)^+ = C'^+$.

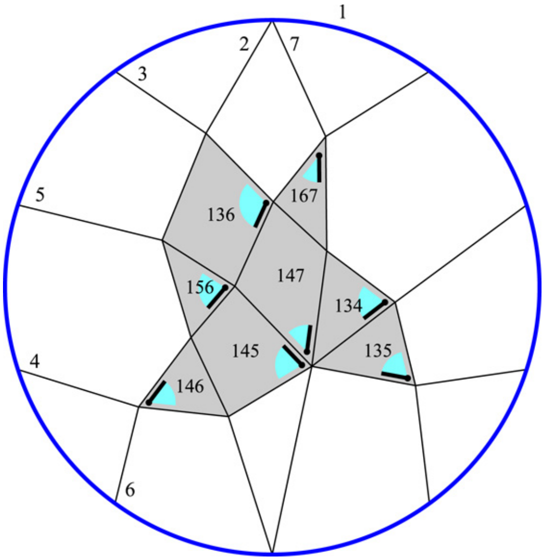


Fig. 7. The active bijection (bounded case).

(5) $(B' \cap D \cap C) - b \cup a \subseteq C^+$
Same proof as that of (4).

By (1)–(5) the hypothesis of the Crescent Lemma is satisfied by B, B' and a, b with the same notation. Hence $B \cap D' \cap C - b = \emptyset$ and $B' \cap D \cap C' - b = \emptyset$. However we have a smallest in $C^*(B; e)$ for some $e \in B \cap D' - b$, or a smallest in $C^*(B'; e)$ for some $e \in B' \cap D - b$. The first case implies $e \in C$ hence $e \in B \cap D' \cap C - b$, and the second implies $e \in B' \cap D \cap C' - b$. We get a final contradiction, proving that $B = B'$. □

The main result of the paper – the existence and unicity of a fully optimal basis in a bounded region – follows easily from Propositions 4.2 and 4.3, by combining with a counting result of Zaslavsky–Las Vergnas.

Theorem 4.5. *Let M be an acyclic oriented matroid on a linearly ordered set with smallest element e_1.*
Then an e_1-bounded region of M has a unique fully optimal basis.
Furthermore, fully optimal bases establish a bijection between the set of e_1-bounded regions of M contained in e_1^+ and the set of uniactive internal bases.

Proof. By the unicity proved in Proposition 4.3, the mapping defined in Proposition 4.2 maps injectively the set of uniactive internal bases of M into the set of e_1-bounded regions contained in e_1^+. By a result of T. Zaslavsky for real arrangements of hyperplanes [24], generalized by M. Las Vergnas to oriented matroids [16], the number of uniactive internal bases is equal to the number of bounded regions contained in e_1^+. Therefore, the mapping of Proposition 4.2 is actually a bijection. Theorem 4.5 follows. □

Definition 4.6. We denote by α_M the bijection of Theorem 4.5 from bounded regions of M to uniactive internal bases, and call it the *active orientation-to-basis bijection*. When M is a bounded acyclic ordered oriented matroid, we denote by $\alpha(M)$ the unique fully optimal basis of its fundamental region. We have $\alpha(-_A M) = \alpha_M(A)$. The active bijection is invariant under reorientation of M with respect to A, up to symmetric difference with A.

The bijection α_M is the bounded case of the active orientation-to-basis mapping.

Fig. 7 shows an example of the active bijection. The sequences of covectors, or flags, considered in the definition of fully optimal bases by adjacency properties are represented in each region by a sequence of nested faces: in rank 3, a vertex, an edge, a region. The active bijection is graphically determined by these flags. The fully optimal basis $\{b_1 = e_1, b_2, b_3\}$ of a region is such that b_2 is the smallest pseudoline containing the vertex of the flag of this region, and b_3 supports the edge of the flag.

We point out that since the feasible region of the pseudolinear program associated with a fully optimal basis is the fundamental region, the feasible region is on the positive side of the objective function independently of its geometric side. Thus, in terms of linear programming, the vertex of a bounded region supporting the fully optimal basis is always a maximum, however the objective function is changed to its opposite when the side changes. This phenomenon is illustrated in Fig. 7, where there are bounded regions on both sides of $f = f_1 = 2$.

The active mapping α_M is extended to general oriented matroids in [11], together with several specializations and variants, such as an activity-preserving bijection between subsets and reorientations, or between no-broken-circuit subsets and acyclic reorientations (regions in a hyperplane arrangement). The mapping α_M can be computed directly by means of pseudolinear programming in the bounded case [10]. It is calculated by deletion/contraction (extending the linear programming classical construction by variable/constraint deletion), and also characterized by some of its properties in [12]. Other particular cases with specific properties are studied in [7–9]. In the uniform case [7], a vertex at finite distance determines uniquely a uniactive internal basis. These bases correspond in a simple way to the optimal vertices of pseudolinear programming.

In the graphical case [8], bounded regions correspond to bipolar orientations. The general active mapping provides a bijection between orientations and edge subsets (or subgraphs), and, in particular, a bijection between acyclic orientations with a given unique sink and increasing spanning trees. In the supersolvable case [9], the active mapping can be derived from a simpler one based on an easy deletion/contraction construction. In particular, it turns out that a well-known bijection of enumerative combinatorics [21, p. 25], between $(n-1)$-permutations and increasing trees on n vertices, is equivalent to the orientation-to-basis mapping applied to the regions of the braid arrangement [9] (see also [8, Sec. 6–7] when applied to a complete graph).

5. Duality

Duality is meaningful in the definition of fully optimal bases, as already seen in Proposition 3.3. We deepen in this Section the relations between fully optimal bases of bounded regions of the oriented matroid and of its dual.

Proposition 5.1. *Let M be an ordered matroid on a set E, and $B_{\min} = \{p, f, \dots\}_<$ be its minimal basis. A basis B of M is internal and uniactive if and only if $(E \setminus B) \setminus \{f\} \cup \{p\}$ is internal and uniactive in M^*.*

Proof. Let B be a uniactive internal basis of M. We have $p \in B$, otherwise p would be the smallest in $C(B; p)$, and hence externally active, contradicting B internal. We have $f \notin B$, otherwise f would be the smallest in $C^*(B; f)$, and hence a second internally active basis element, contradicting B uniactive.

We have $p \in C(B; f)$ otherwise f would be externally active. Hence $B' = B - p + f$ is a basis of M, with $C^*(B; p) = C^*(B'; f)$, $C(B; f) = C(B'; p)$ so p is the minimal element of its fundamental circuit with respect to B'. The fundamental cocircuits of elements of B' are obtained by modular elimination of f between the fundamental cocircuits of B containing f and $C^*(B; p)$. Their smallest elements are either unchanged or replaced with p, so they are never elements of B', hence no element of B' is the minimal element of its fundamental cocircuit with respect to B'. All the same, the smallest elements of fundamental circuits of elements of $B' - f$ are unchanged and belong to B', so only p is the minimal element of its fundamental circuit with respect to B'. Since the fundamental circuit, resp. cocircuit, of an element with respect to a basis in M equals the fundamental cocircuit, resp. circuit, of this element with respect to the complementary basis in M^*, we have proved that $(E \setminus B) \setminus f \cup p$ is a uniactive internal basis of M^*

The converse implication is deduced by duality. $\square$

Proposition 5.2. *An oriented matroid M on E is acyclic and p-bounded for $p \in E$ if and only $-_pM^*$ is acyclic and p-bounded.*

Proof. It suffices to show one implication, the other follows by duality. Let M be a p-bounded acyclic oriented matroid M. Then M one maximal covector is positive — M acyclic, and every positive cocircuit of M contains p — M is p-bounded. If there exists a positive cocircuit of $-_pM$, then it contains p otherwise it would be a positive cocircuit of M not containing p, so it is a cocircuit D of M with only negative element p. On the other hand M has a positive cocircuit D' with smallest element p. The elimination of p between D and D' gives a positive cocircuit of M not containing p which is a contradiction. So $-_pM$ is totally cyclic.

By hypothesis $-_pM$ has a maximal covector with only negative element p. Let C be a positive circuit C of $-_pM$, then it contains p by orthogonality with the previous maximal covector, so every positive circuit of $-_pM$ contains p. That is, by definition, $-_pM^*$ is acyclic and bounded. □

Proposition 5.1 defines a bijection between uniactive internal bases of M and uniactive internal bases of M^*. Proposition 5.2 defines a bijection between bounded regions of M and bounded regions of M^*. The next proposition states that the active bijection is compatible with these two bijections.

Theorem 5.3. *Let M be a bounded acyclic ordered oriented matroid, with minimal basis $B_{\min} = \{p, f, \ldots\}_<$. We have*

$$\alpha(M) = \big(E \setminus \alpha(-_pM^*)\big) \setminus \{f\} \cup \{p\}.$$

Proof. For M an acyclic bounded ordered oriented matroid, let $B = \alpha(M)$ and $B' = B \setminus p \cup f$. According to Proposition 5.1, $E \setminus B'$ is uniactive internal in M^*. On the other hand, according to Proposition 5.2, $-_pM^*$ is acyclic and bounded.

The fundamental cocircuits of B' are obtained by modular elimination (which is unique) of f between $C^*(B; p) = C^*(B'; f) = D$ and $C^*(B; b)$, $b \in B - p$. Thus $C^*(B; b)$ and $C^*(B'; b)$ have same elements not belonging to D. Hence, the first algorithm of Proposition 4.2 for B in M and the second algorithm of Proposition 4.2 for $E \setminus B'$ in $-_pM^*$ define exactly the same sequence of covectors of M. Hence $E \setminus B' = \alpha(-_pM^*)$. □

We call the duality property in Theorem 5.3 the *active duality* to distinguish it from the simpler *basis duality* defined by $\alpha(M^*) = E \setminus \alpha(M)$ (see [11]).

It has been noticed after Proposition 3.3 that a fully optimal basis B is determined by its flag $C^*_M(B; b_1) \circ C^*_M(B; b_2) \circ \cdots \circ C^*_M(B; b_i)$, $1 \le i \le r$, formed by positive covectors in the associated region. As noticed in the proof of Theorem 5.3, exchanging $p = b_1 = e_1$ and f in B does not change the supports of these covectors. Hence the two sets of regions, one in M, the other in $-_pM^*$ considered about Proposition 3.3 are just exchanged when p and f are exchanged and M and $-_pM^*$ are exchanged. This gives exactly the active duality property.

The active duality can be considered as a strengthening of duality in linear programming. An oriented matroid program $\mathcal{P} = (M; p, f)$ is equivalent to a dual oriented matroid program $\mathcal{P}^* = (M^*, f, p)$ [1, Chap. 10]. More precisely, if B is an optimal basis of $\mathcal{P}$ on E, then $E \setminus B$ is an optimal basis of the dual program $\mathcal{P}^*$. Actually, the tableau of $E \setminus B$ is essentially obtained by transposing and negating the tableau of B, and the Simplex Criterion for B is equivalent to the Simplex Criterion for $E \setminus B$, with the variation that the feasible region is on the negative side of the objective function, and thus its sign in the fundamental cocircuit of the hyperplane at infinity is negative and the sign of the hyperplane at infinity in its fundamental circuit is positive ([1, Cor. 10.2.9]). Theorem 5.3 expresses that active duality for a fully optimal basis amounts to transpose and negates its tableau, and exchange p and f in the basis to consider a uniactive internal one. Notice that p and f appear to play here symmetric parts, which is also the case, with a slight variation, in the dual program $\mathcal{P}^* = (M^*, f, p)$ where p and f exchange their respective parts of hyperplane at infinity and objective function. Anyway, the fundamental cocircuit of p resp. circuit of f with respect to $\alpha(M)$ in M equals the fundamental circuit of f resp. cocircuit of p with respect to $\alpha(-_pM^*)$ in M^* (exchanging p and f in the basis does not change the first fundamental circuit and cocircuit). Hence, according to the definitions, the fully optimal basis of $-_p M^*$ is an optimal basis of the dual program $\mathcal{P}^* = (M^*, f, p)$. Thus, the active duality property refines pseudolinear programming duality, see also [10].

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler, Oriented Matroids, 2nd ed., in: Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, UK, 1999.
[2] J. Bokowski, Computational Oriented Matroids, Cambridge University Press, Cambridge, UK, 2006.
[3] T. Brylawski, D. Lucas, Uniquely representable combinatorial geometries, in: B. Segre (Ed.), Teorie Combinatorie, Accademia Nazionale dei Lincei, Roma, Italy, 1976, pp. 83–108.
[4] T. Brylawski, J. Oxley, The Tutte polynomial and its applications, in: N. White (Ed.), Matroid Applications, Cambridge University Press, Cambridge, UK, 1992, (Chapter 6).
[5] H.H. Crapo, The Tutte polynomial, Aequationes Math. 3 (1969) 211–229.
[6] E. Gioan, Correspondance naturelle entre bases et réorientations des matroïdes orientés, Ph.D. Dissertation, University of Bordeaux 1, December, 2002.
[7] E. Gioan, M. Las Vergnas, Bases, reorientations and linear programming in uniform and rank 3 oriented matroids, Adv. Appl. Math. 32 (2004) 212–238 (special issue Workshop on Tutte polynomials, Barcelona 2001).
[8] E. Gioan, M. Las Vergnas, Activity preserving bijections between spanning trees and orientations in graphs, Discrete Math. 298 (2005) 169–188 (special issue FPSAC 2002).
[9] E. Gioan, M. Las Vergnas, The active bijection between regions and simplices in supersolvable arrangements of hyperplanes, Electronic Journal of Combinatorics 11 (2) (2006) (Special volume dedicated to R. Stanley), #R30.
[10] E. Gioan, M. Las Vergnas, The active bijection in graphs, hyperplane arrangements, and oriented matroids 2. Decomposition of activities (in preparation).
[11] E. Gioan, M. Las Vergnas, The active bijection in graphs, hyperplane arrangements, and oriented matroids 3. Linear programming construction of fully optimal bases (in preparation).
[12] E. Gioan, M. Las Vergnas, The active bijection in graphs, hyperplane arrangements, and oriented matroids 4. Deletion/contraction constructions and universality (in preparation).
[13] E. Gioan, M. Las Vergnas, Fully optimal bases and the active bijection in graphs, hyperplane arrangements, and oriented matroids, (Proceedings EuroComb, Sevilla 2007), Electronic Notes in Discrete Math. 29 (2007) 365–371.
[14] C. Greene, T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions and orientations of graphs, Trans. Amer. Math. Soc.. 280 (1983) 97–126.
[15] M. Las Vergnas, Matroïdes orientables, C. R. Acad. Sci. Paris Sér. A 280 (1975) 61–65.
[16] M. Las Vergnas, Acyclic and totally cyclic orientations of combinatorial geometries, Discrete Math. 20 (1977/78) 51–61.
[17] M. Las Vergnas, Convexity in oriented matroids, J. Combin. Theory Ser. B 29 (1980) 231–243.
[18] M. Las Vergnas, The Tutte polynomial of a morphism of matroids II. Activities of orientations, in: Progress in Graph Theory (Proc. Waterloo Silver Jubilee Conf. 1982), Academic Press, Toronto, Canada, 1984, pp. 367–380.
[19] M. Las Vergnas, A correspondence between spanning trees and orientations in graphs, in: Graph Theory and Combinatorics (Proc. Cambridge Combin. Conf. 1983), Academic Press, London, UK, 1984, pp. 233–238.
[20] R.P. Stanley, Acyclic orientations of graphs, Discrete Math. 5 (1973) 171–178.
[21] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge University Press, Cambridge, UK, 1986.
[22] W.T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.
[23] R.O. Winder, Partitions of N-space by hyperplanes, SIAM J. Appl. Math. 14 (1966) 811–818.
[24] T. Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes, Mem. Amer. Math. Soc. 1 (1) (1975) 154.